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Lagrangian moments and mass transport in Stokes waves Part 2. Water of finite depth

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Some simple relations between the Lagrangian moments and cumulants in a steady finite-amplitude gravity wave on deep water are here generalized to water of finite depth. The first three Lagrangian moments are shown to be given in term of the mass-transport velocity U at the free surface, the potential and kinetic energy densities V and T, and the mean-square particle velocity $\overline{u_B^2}$ on the bottom.

A simple method of calculation is described, which exploits certain quadratic relations between the Fourier coefficients in Stokes's series. The ratio U/c and the associated Lagrangian skewness is calculated for periodic waves, as a function of the wave steepness and the mean water depth.

For limiting waves, i.e. those with sharp crests, it is found that the most symmetric orbits, in the Lagrangian sense, occur not in very deep or very shallow water, but at one intermediate value of the ratio of depth to wavelength. When the depth parameter kd equals 1.93 the vertical displacement of a marked particle at the free surface is closely sinusoidal in the time t.

1. Introduction

Lagrangian methods of observing ocean waves, by photographing a floating marker, or by means of an accelerometer mounted in a freely floating buoy, are now commonly employed. However, for waves of finite amplitude, quantities so measured must differ in general from those obtained with fixed instruments. In two recent papers (1986*a*, 1987) the present author pointed out some simple but exact relations between the Lagrangian surface elevation – particularly its mean value, variance and skewness – and other integral properties of the wave, namely the potential and kinetic energies V and T, the phase-speed c and the mass-transport velocity U at the free surface. The analysis was for deep-water waves, with particular application to remote sensing from satellites.

In shallow water the mean depth of water is sometimes measured by observing photographically the vertical displacements of a floating body. The present paper generalizes the previous relations for deep water to Stokes waves in water of any finite depth. The analysis is only slightly longer, and the final results (see §9) are found to be formally almost as simple. Extensive use is made of certain quadratic relations between Stokes coefficients (see Longuet-Higgins 1978) for waves in finite depth, and a computational method based on these relations is described and carried out.

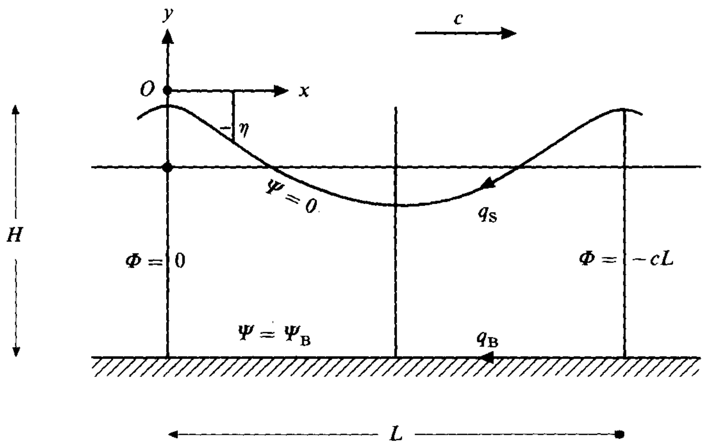

FIGURE 1. Sketch defining coordinates and parameters for waves in water of finite depth.

2. Notation and definitions

Consider steady, irrotational waves of finite amplitude a, travelling to the right with speed c, as in figure 1. To fix a reference frame, suppose first that the time-mean value $\bar{u}$ of the horizontal fluid velocity u at some fixed point beneath the level of the wave troughs vanishes. Then, since the motion is irrotational, $\bar{u}$ vanishes at all such points. In particular, if u_B is the horizontal velocity at the bottom,

$$\bar{u}_B = 0. \tag{2.1}$$

The phase-speed c is measured relative to this reference frame.

Now relative to another reference frame moving with horizontal speed c the motion becomes steady, and we may fix the origin of the vertical coordinate y by supposing that in Bernoulli's equation

$$\frac{p}{\rho} + \tfrac{1}{2}q^2 + gy = \text{constant} \tag{2.2}$$

the constant on the right-hand side is zero. Here p, q and ρ denote the pressure, particle speed and the fluid density, and g is gravity. The origin O is then somewhat above the crest level, as in figure 1. We shall not at first assume that origin lies vertically above a wave crest, or indeed that the free surface is symmetric fore-and-aft. In this reference frame the bottom will be given by $y = -H$, say, the free surface by $y = \eta$, and the (Eulerian) mean level by

$$\bar{\eta}_E = h - H, \tag{2.3}$$

where h is the mean depth of water.

Now if q_S and q_B denote the values of q at the free surface ($p = 0$) and at the bottom respectively we have from (2.2)

$$\tfrac{1}{2}q_S^2 + g\eta = 0 \tag{2.4}$$

and

$$\frac{p_B}{\rho} + \tfrac{1}{2}q_B^2 - gH = 0. \tag{2.5}$$

On taking mean values with respect to x (or time t) we have

$$\frac{\bar{p}_B}{\rho} = gH - \tfrac{1}{2}\overline{q_B^2}. \tag{2.6}$$

But since on the bottom the vertical component of velocity vanishes there is no vertical flux of momentum there. Hence

$$\frac{\bar{p}_B}{\rho} = gh. \tag{2.7}$$

From (2.3), (2.6) and (2.7) it follows that

$$g\bar{\eta}_E = g(h-H) = -\tfrac{1}{2}\overline{q_B^2} \tag{2.8}$$

so
$$\bar{\eta}_E = -\frac{1}{2g}\overline{q_B^2}. \tag{2.9}$$

Since $q_B = c - u_B$ it follows from (2.9) and (2.1) that

$$\bar{\eta}_E = -\frac{1}{2g}(c^2 + \bar{u}_B^2). \tag{2.10}$$

This is the appropriate generalization of the deep-water case (Longuet-Higgins 1986*a*, equation (3.13)) when u_B is not zero.

3. Lagrangian mean values

General formulae for the Lagrangian moments can be derived by the same argument as in Longuet-Higgins (1987). Thus the rth Lagrangian moment of η is by definition

$$m_{Lr} = \frac{1}{T_L}\int \eta^r \,\mathrm{d}t, \tag{3.1}$$

where T_L denotes the Lagrangian period:

$$T_L = \int \mathrm{d}t \tag{3.2}$$

and t denotes the time following a particle. But if Φ denotes the velocity potential in the moving reference frame we have

$$\mathrm{d}t = q^{-2}\,\mathrm{d}\Phi = -\frac{1}{2g\eta}\,\mathrm{d}\Phi \tag{3.3}$$

by (2.4). So from (3.1)

$$m_{Lr} = -\frac{1}{2gT_L}\int \eta^{r-1}\,\mathrm{d}\Phi. \tag{3.4}$$

In the case $r = 1$ this gives for the Lagrangian-mean level

$$\bar{\eta}_L = -\frac{1}{2gT_L}\int \mathrm{d}\Phi, \tag{3.5}$$

and since
$$\int \mathrm{d}\Phi = \int \Phi_x\,\mathrm{d}x = (c-\bar{u})\,L = cL \tag{3.6}$$

it follows that
$$\bar{\eta}_L = -\frac{cL}{2gT_L} = -\frac{c^2}{2g}\frac{T_E}{T_L}, \tag{3.7}$$

$T_E = L/c$ being the Eulerian wave period. From (3.7) and (2.10) we have

$$\bar{\eta}_L - \bar{\eta}_E = \frac{1}{2g}\left[c^2\left(1-\frac{T_E}{T_L}\right) + \bar{u}_B^2\right]. \tag{3.8}$$

But, quite generally,

$$\frac{T_{\mathrm{E}}}{T_{\mathrm{L}}}=\frac{L/c}{L/(c-U)}=1-\frac{U}{c} \tag{3.9}$$

so that (3.8) can also be written

$$\bar{\eta}_{\mathrm{L}}-\bar{\eta}_{\mathrm{E}}=\frac{1}{2g}(Uc+\bar{u}_{\mathrm{B}}^{2}). \tag{3.10}$$

This is the appropriate generalization of Longuet-Higgins (1986*a*, equation (3.8)) when $u_{\mathrm{B}} \neq 0$.

4. Higher moments

To proceed further we introduce the stream function Ψ of the steady motion as seen in the moving reference frame, and take $\Psi=0$ at the free surface, $\Psi=\Psi_{\mathrm{B}}$ on the bottom. It is convenient to take first the case when the waves are symmetric about the crests, $\Phi=0$ say, then the coordinates X, Y in the moving frame can be expressed in terms of Φ, Ψ by the Fourier series

$$\left.\begin{aligned} Y+\frac{\Psi}{c}&=\tfrac{1}{2}A_0+\sum_{n=1}^{\infty}A_n\sinh\frac{n(\Psi_{\mathrm{B}}-\Psi)}{c}\cos\frac{n\Phi}{c}\\ X+\frac{\Phi}{c}&=-\sum_{n=1}^{\infty}A_n\cosh\frac{n(\Psi_{\mathrm{B}}-\Psi)}{c}\sin\frac{n\Phi}{c}\end{aligned}\right\} \tag{4.1}$$

the A_n being real constants, to be determined. The equation of the free surface $\Psi=0$ can therefore be expressed parametrically in the form

$$\left.\begin{aligned} \eta&=\tfrac{1}{2}a_0+\sum_{1}^{\infty}a_n\cos n\theta,\\ X&=\gamma_0\theta+\sum_{1}^{\infty}a_n\gamma_n\sin n\theta,\end{aligned}\right\} \tag{4.2}$$

where we have written $\theta=-\Phi/c$,

$$a_0=A_0,\quad a_n=A_n\sinh\frac{n\Psi_{\mathrm{B}}}{c},\quad n=1,2,\ldots \tag{4.3}$$

and

$$\gamma_0=1,\quad \gamma_n=\coth\frac{n\Psi_{\mathrm{B}}}{c},\quad n=1,2,\ldots. \tag{4.4}$$

Note that a_0 is generally negative, but that γ_n, like Ψ_{B}, is positive. It will be helpful to write also

$$b_0=1,\quad b_n=na_n,\quad n=1,2,\ldots \tag{4.5}$$

and define

$$\left.\begin{aligned} J&=\tfrac{1}{2}(a_1^2+a_2^2+\ldots),\\ K&=\tfrac{1}{2}(a_1b_1\gamma_1+a_2b_2\gamma_2+\ldots),\\ N&=\tfrac{1}{4}[b_1^2(1+\gamma_1^2)+b_2^2(1+\gamma_2^2)+\ldots].\end{aligned}\right\} \tag{4.6}$$

Note that the Eulerian mean level is given immediately by

$$\bar{\eta}_{\mathrm{E}}=\frac{1}{L}\int\eta\,\mathrm{d}X=\tfrac{1}{2}a_0+K, \tag{4.7}$$

so that from (2.10) we have also

$$\bar{u}_{\mathrm{B}}^{2}=-c^2-g(a_0+2K). \tag{4.8}$$

For the second Lagrangian moment we have, on setting $r = 2$ in (3.4),

$$m_{L2} = -\frac{1}{2gT_L}\int \eta \,\mathrm{d}\Phi = -\frac{1}{2gT_L}\tfrac{1}{2}a_0 cL = -\frac{c^2}{2g}\frac{T_E}{T_L}(\tfrac{1}{2}a_0) \tag{4.9}$$

and similarly for the third moment

$$m_{L3} = -\frac{1}{2gT_L}\int \eta^2 \,\mathrm{d}\Phi = -\frac{cL}{2gT_L}(\tfrac{1}{4}a_0^2+J) = -\frac{c^2}{2g}\frac{T_E}{T_L}(\tfrac{1}{4}a_0^2+J). \tag{4.10}$$

These expressions will now be related to other integral properties of the motion.

5. Momentum and kinetic energy densities

Let
$$I = \frac{1}{L}\int_0^L\int_{-H}^{\eta} u \,\mathrm{d}x\,\mathrm{d}y \tag{5.1}$$

denote the mean density of the impulse, or horizontal momentum of the wave, in the original frame of reference. If $\phi = \Phi + cX$ denotes the velocity potential in this reference frame it may be shown that

$$I = \frac{1}{L}\int (\eta - \bar{\eta}_E)\,\mathrm{d}\phi \tag{5.2}$$

(see Longuet-Higgins 1975, equation (2.5)). Since $\mathrm{d}\phi = \mathrm{d}\Phi + c\mathrm{d}X$ this becomes

$$I = \frac{1}{L}\int (\eta - \bar{\eta}_E)\,\mathrm{d}\Phi. \tag{5.3}$$

Substitution from (4.2) then gives

$$I = -c(\tfrac{1}{2}a_0 - K) = cK \tag{5.4}$$

by (4.7).

Now the kinetic energy density T, defined by

$$T = \frac{1}{L}\int_0^L\int_{-H}^{\eta} \tfrac{1}{2}(\phi_x^2 + \phi_y^2)\,\mathrm{d}x\,\mathrm{d}y \tag{5.5}$$

is related to the momentum density I by

$$2T = cI \tag{5.6}$$

(see for example Longuet-Higgins 1975, equation (B)). So fom (5.4) and (5.6) we have

$$2T = c^2 K, \tag{5.7}$$

where K is given by (4.6).

6. The potential energy

The potential energy density V is related to the second Eulerian moment

$$m_{E2} = \frac{1}{L}\int \eta^2 \,\mathrm{d}X. \tag{6.1}$$

Substitution from the series (4.2) gives

$$m_{E2} = \tfrac{1}{4}a_0^2 + J + a_0 K + R, \tag{6.2}$$

where
$$R = \frac{1}{L}\int \sum_1^\infty a_l \cos l\theta \sum_1^\infty a_m \cos m\theta \sum_1^\infty b_n \gamma_n \cos n\theta \,\mathrm{d}\theta. \tag{6.3}$$

Carrying out the integration we obtain

$$\begin{aligned} 4R = {} & b_1\gamma_1 \quad [2(a_1a_2 + a_2a_3 + \ldots] \\ & + b_2\gamma_2[2(a_1a_3 + a_2a_4 + \ldots) + (a_1a_1)] \\ & + b_3\gamma_3[2(a_1a_4 + a_2a_5 + \ldots) + (a_1a_2 + a_2a_1)] \\ & + b_4\gamma_4[2(a_1a_5 + a_2a_6 + \ldots) + (a_1a_3 + a_2a_2 + a_3a_1)] \\ & + \ldots . \end{aligned} \tag{6.4}$$

To reduce this expression we make use of the integral relation

$$\int_0^{cL} Y(X_\Phi + \mathrm{i}\gamma_m Y_\Phi)\,\mathrm{e}^{\mathrm{i}m\Phi/c}\,\mathrm{d}\Phi = 0, \tag{6.5}$$

where m is any positive integer (see Longuet-Higgins 1978, equation (7.8)). Substitution from (4.2) yields the following system:

$$\sum_{n=0}^{m} a_{m-n} b_n \gamma_n + \sum_{n=m+1}^{\infty} a_{n-m} b_n \gamma_n + \sum_{n=0}^{\infty} a_{m+n} b_n \gamma_n + \left[\sum_{n=0}^{m} a_{m-n} b_n + \sum_{n=m+1}^{\infty} a_{n-m} b_n - \sum_{n=0}^{\infty} a_{m+n} b_n\right]\gamma_m = 0, \tag{6.6}$$

with
$$m = 1, 2, \ldots . \tag{6.7}$$

Multiplying (6.6) by a_m and summing over the positive integers m we obtain

$$(\alpha + \bar{\alpha}) + (\beta + \bar{\beta}) + (\gamma - \bar{\gamma}) + 4J + 4a_0 K = 0, \tag{6.8}$$

where
$$\left.\begin{aligned} \alpha &= \sum_{m=2}^{\infty} a_m \sum_{n=1}^{m-1} a_{m-n} b_n \gamma_n, \\ \bar{\alpha} &= \sum_{m=2}^{\infty} a_m \sum_{n=1}^{\infty} a_{m-n} b_n \gamma_m, \end{aligned}\right\} \tag{6.9}$$

$$\left.\begin{aligned} \beta &= \sum_{m=1}^{\infty} a_m \sum_{n=m+1}^{\infty} a_{n-m} b_n \gamma_n, \\ \bar{\beta} &= \sum_{m=1}^{\infty} a_m \sum_{n=m+1}^{\infty} a_{n-m} b_n \gamma_m, \end{aligned}\right\} \tag{6.10}$$

$$\left.\begin{aligned} \gamma &= \sum_{m=1}^{\infty} a_m \sum_{n=1}^{\infty} a_{m+n} b_n \gamma_n, \\ \bar{\gamma} &= \sum_{m=1}^{\infty} a_m \sum_{n=1}^{\infty} a_{m+n} b_n \gamma_m. \end{aligned}\right\} \tag{6.11}$$

The following identities are easily established:

$$\beta = 2\bar{\alpha}, \quad \bar{\beta} = \alpha + \bar{\gamma}, \quad \gamma = \alpha. \tag{6.12}$$

From (6.8) it now follows that

$$\alpha + \bar{\alpha} = -\tfrac{4}{3}(J + a_0 K). \tag{6.13}$$

But (6.4) can be written
$$4R = 2(\alpha+\bar{\alpha}). \tag{6.14}$$
It follows that
$$R = -\tfrac{2}{3}(J+a_0 K) \tag{6.15}$$
and so from (6.2)
$$m_{\mathrm{E}2} = \tfrac{1}{4}a_0^2+\tfrac{1}{3}(J+a_0 K). \tag{6.16}$$
So the potential energy density V is given by
$$2V/g = m_{\mathrm{E}2}-\bar{\eta}_{\mathrm{E}}^2 \tag{6.17}$$
$$= \tfrac{1}{3}(J-2a_0 K-3K^2) \tag{6.18}$$
from (4.7). Alternative expressions are
$$2V/g = \tfrac{1}{3}(J-4K\bar{\eta}_{\mathrm{E}}+K^2) \tag{6.19}$$
or
$$\frac{2V}{g} = \frac{1}{3}\left[J+\frac{2K(c^2+\overline{u_{\mathrm{B}}^2})}{g}+K^2\right] \tag{6.20}$$
by (2.10). In deep water u_{B} vanishes and (6.12) reduces to (6.2) of Longuet-Higgins (1984), after correction of a misprint.

7. The Lagrangian density $(T-V)$

By inspection, the system of equations (6.6) may be expressed in the form
$$\frac{\partial F}{\partial a_m} = 0, \quad m = 1,2,\ldots, \tag{7.1}$$
where
$$\begin{aligned} F &= F(a_0, a_1, a_2, \ldots; c^2, \overline{u_{\mathrm{B}}^2}), \\ &= (J+a_0 K)+\tfrac{1}{2}(\alpha+\bar{\alpha})+\tfrac{1}{4}(\alpha_0+c^2+\overline{u_{\mathrm{B}}^2})^2. \end{aligned} \tag{7.2)}$$
Since α and $\bar{\alpha}$ do not involve a_0 explicitly we have also
$$\frac{\partial F}{\partial a_0} = K+\tfrac{1}{2}(a_0+c^2+\overline{u_{\mathrm{B}}^2}) = 0 \tag{7.3}$$
by (4.8). Thus the total differential $\mathrm{d}F$ satisfies
$$\begin{aligned} \mathrm{d}F &= \sum_{m=0}^{\infty} \frac{\partial F}{\partial a_m}\,\mathrm{d}a_m+\frac{\partial F}{\partial c^2}\,\mathrm{d}c^2+\frac{\partial F}{\partial \overline{u_{\mathrm{B}}^2}}\,\mathrm{d}\overline{u_{\mathrm{B}}^2}, \\ &= \tfrac{1}{2}(a_0+c^2+\overline{u_{\mathrm{B}}^2})\,(\mathrm{d}c^2+\mathrm{d}\overline{u_{\mathrm{B}}^2}), \\ &= -K(\mathrm{d}c^2+\mathrm{d}\overline{u_{\mathrm{B}}^2}). \end{aligned} \tag{7.4}$$

The function F is related to the Lagrangian density $\mathscr{L} = (T-V)$ as follows. From (5.7) and (6.19) we have (if $g = 1$).
$$-2\mathscr{L} = \tfrac{1}{3}(J-4K\bar{\eta}_{\mathrm{E}}+K^2)-c^2K \tag{7.5}$$
and on substitution for $\bar{\eta}$ and c^2 from (4.7) and (4.8) we find
$$-2\mathscr{L} = \tfrac{1}{3}(J+a_0 K)+K(K+\overline{u_{\mathrm{B}}^2}). \tag{7.6}$$
On the other hand from (7.2), (6.13) and (4.8) we have
$$F = \tfrac{1}{3}(J+a_0 K)+K^2. \tag{7.7}$$

Hence
$$F = -2\mathscr{L} - K\overline{u_B^2}. \tag{7.8}$$
This generalizes to finite depth the simpler relation $F = -2\mathscr{L}$ valid when $u_B = 0$ (see Longuet-Higgins 1985, equation (3.7)).

From (7.8) it follows also that
$$2\mathrm{d}\mathscr{L} = -\mathrm{d}F - \overline{u_B^2}\,\mathrm{d}K - K\,\mathrm{d}\overline{u_B^2} \tag{7.9}$$
$$= K\,\mathrm{d}c^2 - \overline{u_B^2}\,\mathrm{d}K \tag{7.10}$$
by (7.4). This can be reconciled with a more general relation (Longuet-Higgins 1975, equation (4.17)) namely
$$\mathrm{d}\mathscr{L} = 2T\frac{\mathrm{d}c}{c} + (T-2V+Bh)\frac{\mathrm{d}L}{L} - \tfrac{1}{2}\overline{u_B^2}\,\mathrm{d}h. \tag{7.11}$$
For, we have $2T\,\mathrm{d}c/c = T\,\mathrm{d}c^2/c^2 = \frac{1}{2}Kc^2$, and since the wavelength L is fixed, $\mathrm{d}L$ vanishes. Thus (7.11) becomes
$$2\mathrm{d}\mathscr{L} = K\,\mathrm{d}c^2 - \overline{u_B^2}\,\mathrm{d}h. \tag{7.12}$$
But in the differentiation of (7.2) the coefficients γ_m were kept constant, implying that Q is constant, where
$$Q = \frac{\Psi_B}{c} = h - \frac{I}{c} = h - K. \tag{7.13}$$
Thus $\mathrm{d}h = \mathrm{d}K$ and (7.10) and (7.12) are seen to be equivalent.

We note that (7.1) and (7.2) provide an alternative proof of (6.13). For, J and K are each homogenous quantities of degree 2 in $a_{1,2}, \ldots$. Also $\alpha, \bar{\alpha}$ are both homogenous quantities of degree 3. So by Euler's theorem
$$\sum_{m=1}^{\infty} a_m \frac{\partial F}{\partial a_m} = 2(J + a_0 K) + \tfrac{3}{2}(\alpha + \bar{\alpha}). \tag{7.14}$$
But the left-hand side vanishes, by (7.1), so that (6.13) follows immediately.

8. The mass-transport velocity

From (3.2) and (3.3) of the present paper we have
$$T_L = \int q^{-2}\,\mathrm{d}\Phi = \int (X_\phi^2 + Y_\phi^2)\,\mathrm{d}\Phi = \frac{L}{c}(1+2N), \tag{8.1}$$
where N is given by (4.6). Since $L/c = T_E$ we obtain
$$\frac{T_L}{T_E} = 1 + 2N, \tag{8.2}$$
hence
$$\frac{U}{c} = 1 - (1+2N)^{-1} \tag{8.3}$$
as in the deep-water case, but with N given by the more general expression (4.6).

9. Cumulants of η_L

We may summarize previous results for the moments of η_L by writing
$$A = -\tfrac{1}{2}a_0, \quad B = \frac{T_E}{T_L}, \quad C = \frac{c^2}{2g} \tag{9.1}$$

so (3.7), (4.9) and (4.10) become respectively

$$\left.\begin{aligned} m_{L1} &= -BC, \\ m_{L2} &= BCA, \\ m_{L3} &= -BC(A^2+J). \end{aligned}\right\} \tag{9.2}$$

The second and third cumulants can then be written

$$\kappa_{L2} = m_{L2} - m_{L1}^2 = BC(A-BC) \tag{9.3}$$

and

$$\begin{aligned} \kappa_{L3} &= m_{L3} - 3m_{L2}\,m_{L1} + 2m_{L1}^3 \\ &= -BC[(A-BC)(A-2BC)+J]. \end{aligned} \tag{9.4}$$

In terms of the physical quantities T, V, c^2 and $\overline{u_B^2}$ we have from (5.7)

$$K = \frac{2T}{c^2}. \tag{9.5}$$

Then from (2.10) and (4.7)

$$A = \frac{2T}{c^2} + \frac{c^2 + \overline{u_B^2}}{2g} \tag{9.6}$$

and from (6.20)

$$J = 6V/g - 4T^2/c^4 - 4T(1+\overline{u_B^2}/c^2)/g \tag{9.7}$$

while from (7.1)

$$B = 1 - \frac{U}{c}. \tag{9.8}$$

In deep water, when $\overline{u_B^2}/c^2$ vanishes, these expressions reduce formally to those given in §4 of Longuet-Higgins (1987).

10. Computation

To obtain numerical values for waves in water of moderate depth, we may first normalize with respect to the wavelength L by choosing $L = 2\pi$, and also $g = 1$. We then choose a given value of Q/c (equation (7.13)), enabling us to compute the coefficients γ_m (see (4.4)) as far as required. The system of equations (6.6) can then be solved for $a_0, a_1, a_2, \ldots$ subject to a given amplitude

$$a_1 + a_3 + a_5 + \ldots = a \tag{10.1}$$

say, by the method in §5 of Longuet-Higgins (1985), for waves in deep water. An alternative parameter is

$$1 + \tfrac{1}{2}a_0 + a_1 + a_2 + \ldots = 1 - \tfrac{1}{2}q_{\text{crest}}^2. \tag{10.2}$$

We note that since (6.6) involves only positive values of m, the system can be expressed in the more convenient form

$$F_m \equiv \frac{\partial F^*}{\partial a_m} = 0, \quad m = 1, 2, \ldots, \tag{10.3}$$

where F^* is an abbreviated form of F, namely

$$F^* = (J + a_0 K) + \tfrac{1}{2}(\alpha + \bar{\alpha}). \tag{10.4}$$

The Jacobian of this system, namely

$$\left(\frac{\partial F_m}{\partial a_l}\right) = \left(\frac{\partial^2 F^*}{\partial a_m\, \partial a_l}\right), \tag{10.5}$$

is obviously symmetric. In fact we find

$$2\frac{\partial F_m}{\partial a_l} = (\epsilon_l + \epsilon_m + \epsilon_{l+m})\, a_{l+m} + (\epsilon_l + \epsilon_m + \epsilon_{|l-m|})\, a_{|l-m|} + 2\delta_{l,\,m}, \tag{10.6}$$

where we have written

$$\epsilon_0 = 0, \quad \epsilon_n = n\gamma_n, \quad n = 1, 2, \dots \tag{10.7}$$

and where $\delta_{l,\,m} = 1$ when $l = m$, and 0 otherwise. Moreover, when $l = 0$ and $m > 0$ we have

$$2\frac{\partial F_m}{\partial a_0} = \epsilon_m\, a_m. \tag{10.8}$$

In deep water these equations reduce to the form

$$\frac{\partial F_m}{\partial a_l} = (l+m)\, a_{l+m} + \max\,(l, m)\, a_{|l-m|} + \delta_{l,\,m} \tag{10.9}$$

except that when $l = 0$, $m > 0$,

$$2\frac{\partial F_m}{\partial a_0} = a_m \tag{10.10}$$

(cf. Longuet-Higgins 1985, equation (5.10)).

Having found the coefficients a_m, we can now compute the fundamental quantities J, K and N of §4, (4.6); also the depth h, from

$$h = Q + K \tag{10.11}$$

(see (7.13)), the mean surface level $\bar{\eta}_{\mathrm{E}}$, from (4.7), and the value of $\overline{q_{\mathrm{B}}^2}$, or $(c^2 + \overline{u_{\mathrm{B}}^2})$, from (2.9).

To find the phase speed c, we note that from the series for $\partial X/\partial \Phi$ in (4.11) the fluid velocities q_{c} and q_{t} at the crest ($\theta = 0$) and trough ($\theta = \pi$) are given by

$$\left.\begin{aligned} \frac{c}{q_{\mathrm{c}}} &= -1 - \sum_1^\infty \epsilon_n\, a_n, \\ \frac{c}{q_{\mathrm{t}}} &= -1 - \sum_1^\infty (-)^n\, \epsilon_n\, a_n. \end{aligned}\right\} \tag{10.12}$$

But from (2.4) we have also

$$\left.\begin{aligned} q_{\mathrm{c}}^2 &= -a_0 - 2\sum_1^\infty a_n, \\ q_{\mathrm{t}}^2 &= -a_0 - 2\sum_1^\infty (-)^n\, a_n. \end{aligned}\right\} \tag{10.13}$$

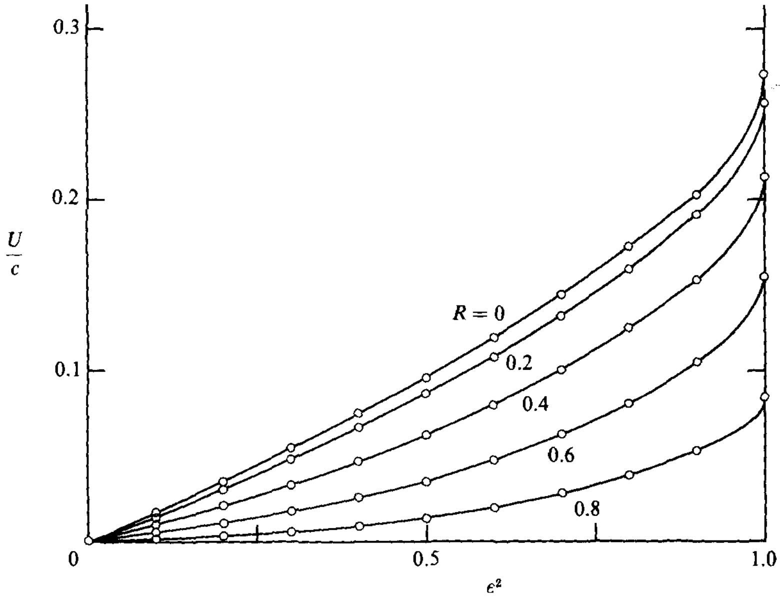

FIGURE 2. The ratio U/c, giving the mass-transport velocity U as a function of ϵ^2, for given values of $R = e^{-Q/c}$.

So we obtain the two alternative expressions

$$\left.\begin{aligned} c^2 &= -\left(a_0 + 2\sum_1^\infty a_n\right)\left(1 + \sum_1^\infty \epsilon_n a_n\right), \\ c^2 &= -\left(a_0 + 2\sum_1^\infty (-)^n a_n\right)\left(1 + \sum_1^\infty (-)^n \epsilon_n a_n\right), \end{aligned}\right\} \tag{10.14}$$

for c^2 in terms of the coefficients a_n. (Of these, the second, which involves alternating series, is likely to be the more accurate.)

The value of T then follows from (5.7), and V and U/c from (6.19) and (8.3) respectively. Hence, with T_E/T_L given by (8.2), we have all the necessary quantities for calculating the moments and cumulants, as in §9.

These computations were carried out for a representative set of parameters, for waves of less than the limiting steepness. For convenience we chose a subset of the parameter values used by Cokelet (1977), who tabulated values of I, T, V and c^2 by an independent method based on cubic relations between the coefficients. Cokelet used regularly spaced values of the parameter

$$\epsilon^2 \equiv 1 - \frac{q_c^2 q_t^2}{c^4} \tag{10.15}$$

which runs monotonically from 0 (for very low waves) to 1 for limiting waves of the same wavelength. The present method can also be adapted to accommodate any specific value of ϵ^2 by noting that

$$\epsilon^2 = 1 - \mu^{-2}, \tag{10.16}$$

where
$$\mu = \frac{c^2}{q_c q_t} = (1 - \epsilon^2)^{-\frac{1}{2}}. \tag{10.17}$$

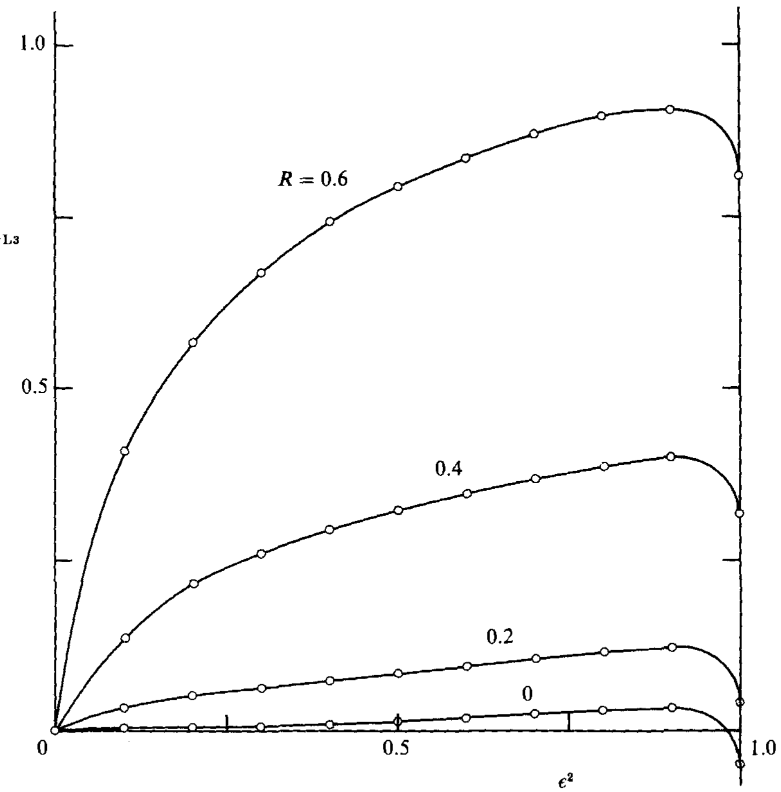

FIGURE 3. The skewness parameter $\lambda_{L3} = \kappa_{L3}/(\kappa_{L2})^{\frac{3}{2}}$, as a function of ϵ^2, for $R = 0$, 0.2, 0.4 and 0.6.

Hence a given value of ϵ^2 corresponds to a given value of μ. But from (10.12) we have

$$\mu = (1+\Sigma)^2 - \Sigma'^2 \tag{10.18}$$

where Σ and Σ' signify the sums of $\epsilon_n a_n$ over even and odd values of n respectively. We may then apply the modification suggested in Longuet-Higgins (1985, §5), noting that

$$\frac{\partial \mu}{\partial a_n} = \begin{Bmatrix} 0 & n = 0, \\ 2\epsilon_n \Sigma & n \text{ even}, \\ -2\epsilon_n \Sigma' & n \text{ odd}. \end{Bmatrix} \tag{10.19}$$

For a sufficient number n_0 of coefficients n, the computed values of a, c^2, T, V, K and $\bar{\eta}_E$ were found to be in agreement with Cokelet's values to at least five significant figures, generally more.

Note that Cokelet's d corresponds to our Q/c. His $\bar{\eta}$ corresponds to our K, and his K corresponds to $(2K - q_B^2)$ in our notation. Like Cokelet we choose as the 'depth' parameter $R = \mathrm{e}^{-d} = \mathrm{e}^{-Q/c}$, which ranges from 0 for deep water to almost 1 for shallow-water waves. The limiting value 1 corresponds to solitary waves.

Figure 2 shows a graph of U/c. We see that, for given R, U/c increases monotonically with the amplitude parameter ϵ^2. Moreover, for given ϵ^2, U/c diminishes monotonically with R. As $R \to 1$, that is in very shallow water, it may be shown that for a given wavelength, U/c must tend to zero asymptotically like $(1-R)$.

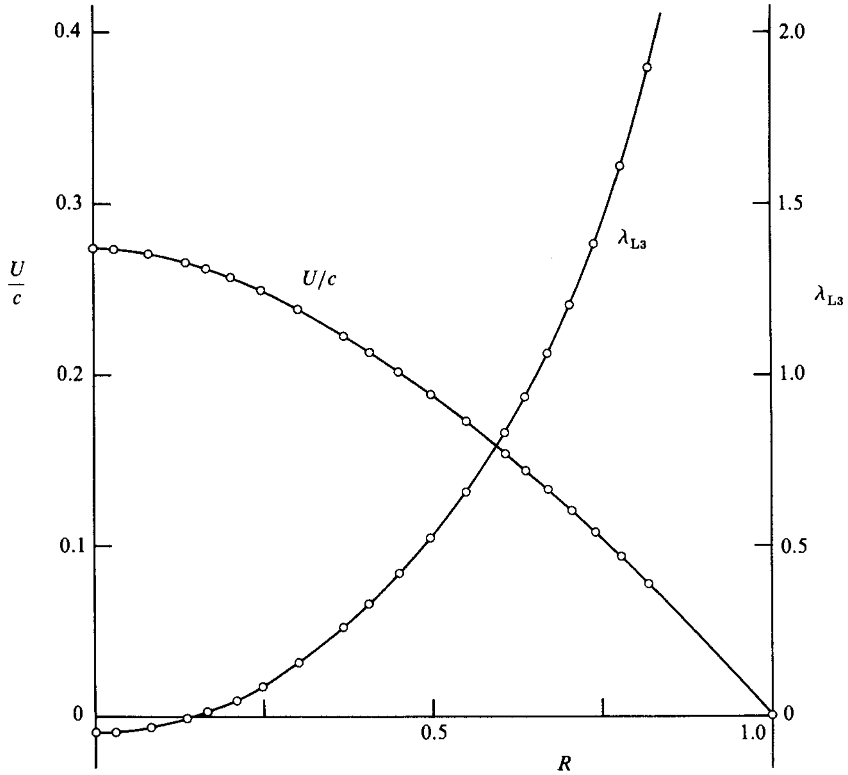

FIGURE 4. U/c and λ_{L3} as functions of R, in waves of limiting steepness ($\epsilon^2 = 1$).

The maximum value $\epsilon^2 = 1$ corresponds to limiting waves, which are best treated by special methods (see §11).

Similarly figure 3 shows the skewness parameter

$$\lambda_{L3} = \frac{\kappa_{L3}}{(\kappa_{L2})^{\frac{3}{2}}} \tag{10.20}$$

plotted against ϵ for chosen values of the depth parameter R. As shown in Longuet-Higgins (1987), the orbital skewness λ_{L3} in deep water ($R = 0$) is remarkably small. As R increases, so $|\lambda_{L3}|$ generally increases with R, at fixed values of ϵ^2, but with an exception in the case of very steep waves. This will now be discussed.

11. Limiting waves ($\epsilon^2 = 1$)

The maximum value $\epsilon^2 = 1$ corresponds to waves of limiting steepness. In this case Fourier series are not convenient, though good results have been achieved with the aid of Padé approximants, as in Cokelet (1977). However, the most accurate calculations are probably those of Williams (1981, 1985) who used special methods appropriate to sharp-crested waves. Williams (1985) tabulates the orbital times $\frac{1}{2}T_L$ for particles at the free surface and elsewhere in the fluid, from which we may calculate

$$\frac{T_L}{T_E} = \frac{cT_L}{L} \tag{11.1}$$

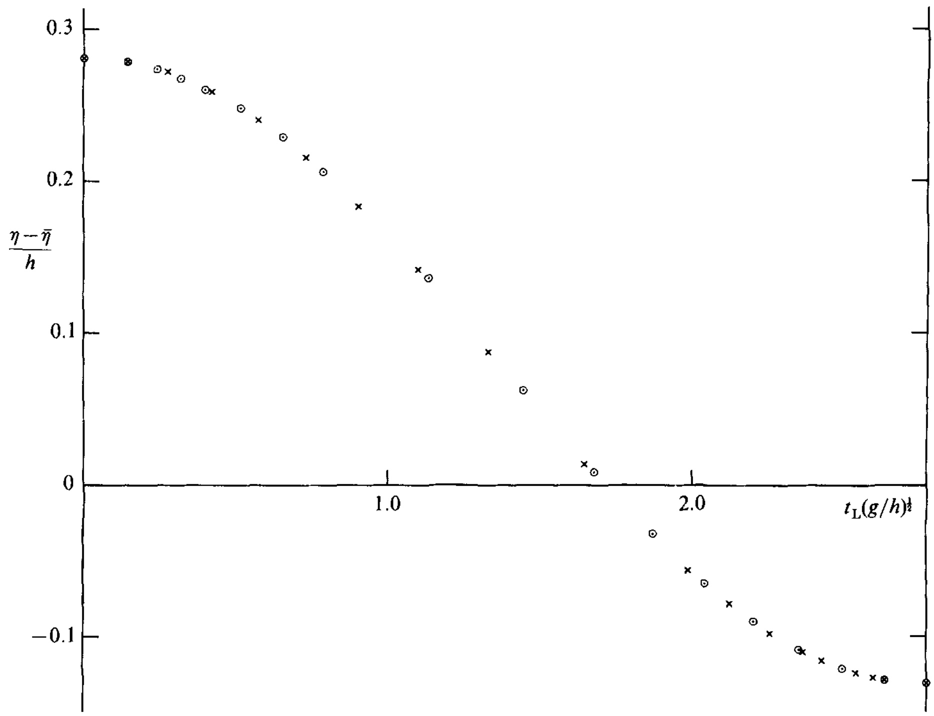

FIGURE 5. The vertical displacement y of a fixed particle at the surface of a limiting wave, for which $d = 2.0$, shown as a function of the orbital time t_L. Circular plots denote original values from Williams (1985). Crosses are obtained by rotation through 180° about the mid-point.

and hence U/c, by (3.9). The results are plotted in figures 2 and 4. It will be seen that U/c decreases monotonically with R. (Williams's calculations were at selected values of $d = \ln(1/R)$.) As $R \to 1$, that is in very shallow water, it may be shown that $U/c \to 0$ linearly with R, that is like $(1-R)$.

As $\epsilon^2 \to 1$ in figure 2 we note a sharp increase in U/C and a vertical tangent corresponding to the behaviour of U in near-limiting waves (see Longuet-Higgins 1986*a*).

There are no tabulated values of the skewness λ_{L3}. However, from the known value of $\overline{U/c}$, and by extracting from Williams (1985) the values of c^2, $\overline{u_B^2}$, V and the 'total head' $\mathscr{H} = h - \bar{\eta}_E$ (where h is the mean depth, also tabulated) we can calculate first

$$A = -\tfrac{1}{2}a_0 = \mathscr{H} - d, \tag{11.2}$$

then B and C from (9.1), then

$$K = -\tfrac{1}{2}(c^2 + \overline{u_B^2} + a_0) \tag{11.3}$$

from (4.8) and

$$J = 6V + 2a_0 K + 3K^2 \tag{11.4}$$

from (6.18). We then have all the quantities necessary to find κ_2 and κ_3 from (9.3) and (9.4), and hence finally λ_{L3}.

The results are shown in figures 3 and 4, second curve (the scale is on the right). It appears that for very steep waves (ϵ^2 near to 1) λ_{L3} has a sharp downturn. This corresponds to the sharp upturn in U in figure 2, which dominates over the oscillatory behaviour in some of the other integral quantities.

In the neighbourhood of $d = 2.0$ ($R = 0.135$) λ_{L3} becomes negative. In other words, as the depth diminishes from infinity, the Lagrangian skewness, which in deep water is already small, diminishes still further before increasing indefinitely as the depth tends to zero ($R \to 1$). The most symmetric orbits, in the Lagrangian sense, occur not in deep water but in water of intermediate depth.

To verify this result we have plotted in figure 5 the vertical displacement y (of surface particles) against the orbital time t when $d = 2.0$, as given by Williams (1985, pp. 474–477). Circular plots are the original values. The crosses represent the same curve rotated through 180° about the mid-point between crest and trough. The two sets of points lie practically on the same curve, showing that the curve is highly symmetric.

Seeing that the flow itself, a steep irrotational wave, is highly nonlinear, this is a curious and unexpected result. The calculated value of λ_{L3} when $d = 2.0$ is 0.0060 (see figure 4). A 6-point interpolation indicates that λ_{L3} vanishes when $d = 1.93$ ($R = 0.145$).

Furthermore, figure 3 suggests that for every value of R less than about 0.1 there exists a small range of very steep waves for which the skewness λ_{L3} is negative or zero.

12. Discussion

As in deep water, the expressions for the Lagrangian moments and cumulants of the surface elevation η are closely related to the mass-transport velocity U, and the expressions for κ_{L2} and κ_{L3} in terms of the integral quantities T, V, c^2 and U/c are formally very similar to those in deep water. The calculations of the Stokes coefficients a_n is also a simple generalization of the corresponding deep-water case. The main difference in finite depth is the role played by the quantity $\overline{u_B^2}$, the mean-square particle velocity at the bottom.

In this paper we have given the analysis appropriate to steady waves of uniform height $2a$. In that case it is known that the wave profile is symmetric so that the coefficients a_n can all be taken as real quantities. The existence of some asymmetric solutions in very nonlinear waves has recently been demonstrated, at least for deep water (Zufiria 1987*b*) and probably also in finite depth (Zufiria 1987*a*). The analysis given in the present paper can easily be generalized to asymmetric motions, by the method used in §8 of Longuet-Higgins (1985). In this way the map of bifurcations of Stokes waves in water of finite depth may be accurately determined.

REFERENCES

COKELET, E. D. 1977 Steep gravity waves in water of arbitrary uniform depth. *Phil. Trans. R. Soc. Lond.* A **286**, 183–230.

LONGUET-HIGGINS, M. S. 1975 Integral properties of periodic gravity waves of finite amplitude. *Proc. R. Soc. Lond.* A **342**, 157–174.

LONGUET-HIGGINS, M. S. 1978 Some new relations between Stokes's coefficients in the theory of gravity waves. *J. Inst. Maths Applics* **22**, 261–273.

LONGUET-HIGGINS, M. S. 1984 New integral relations for gravity waves of finite amplitude. *J. Fluid Mech.* **149**, 205–215.

LONGUET-HIGGINS, M. S. 1985 Bifurcation in gravity waves. *J. Fluid Mech.* **151**, 457–475.

LONGUET-HIGGINS, M. S. 1986*a* Eulerian and Lagrangian aspects of surface waves. *J. Fluid Mech.* **173**, 683–706.

LONGUET-HIGGINS, M. S. 1986*b* Eulerian and Lagrangian wave measurements. In *Proc. Ocean Structural Dynamics Symposium*, pp. 1–32. Corvallis, Ore. Oregon State University. 762 pp.

LONGUET-HIGGINS, M. S. 1987 Lagrangian moments and mass transport in Stokes waves. *J. Fluid Mech.* **179**, 547–555.

WILLIAMS, J. M. 1981 Limiting waves in water of finite depth. *Phil. Trans. R. Soc. Lond.* A **302**, 139-188.

WILLIAMS, J. M. 1985 *Tables of Progressive Gravity Waves*. Pitman. 640 pp.

ZUFIRIA, J. A. 1987*a* Weakly nonlinear non-symmetric gravity waves on water of finite depth. *J. Fluid Mech.* **180**, 371–385.

ZUFIRIA, J. A. 1987*b* Non-symmetric gravity waves on water of infinite depth. *J. Fluid Mech.* **181**, 17–39.